Laplace Transformation of Linear Time-Varying Systems

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Outline of the Presentation

- From LTV Elements to LTV Systems
- Observations on LTV Systems
  - Generalized-Delay System Representation
  - Circularly Symmetric Functions
  - Bivariate Time and Bifrequency Characterization
- Two-Dimensional Laplace Transform (2DLT)
  - The Hankel Transformation
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- An Illustrative Example
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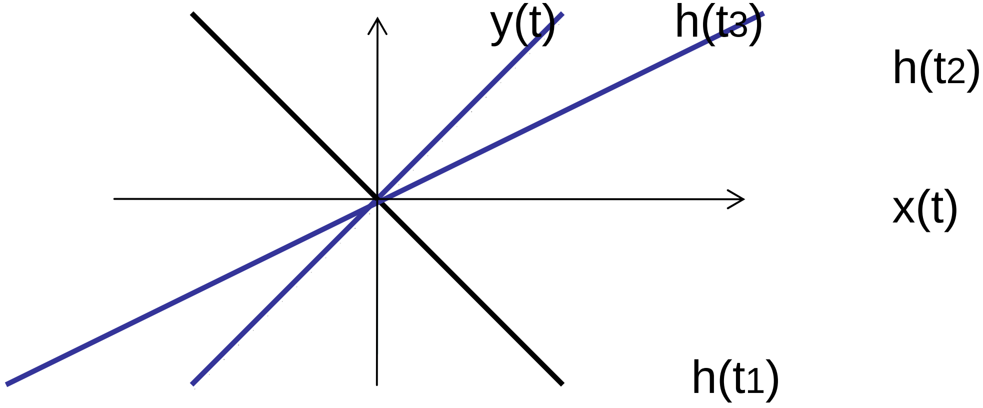
Linear Time Varying Elements

- **A single-input single-output (SISO) dynamic system element of finite order characterized by its input-output relationship is said to be linear if the following holds for each t ≥0:**

$$y(t) = h(t)x(t)$$

  - **Where *h(t)* is the system function defines the response at time t, denotes the slope of the y-x curve in a rectangular coordinates system.**



Linear Time Varying Systems

- **A SISO dynamic system operation is shown symbolically by:**

$$y(t) = o\{x(t)\}$$

- **The system operator is linear if and only if the following relation holds:**

$$o\{\alpha x_1(t) + \beta x_2(t)\} = \alpha o\{x_1(t)\} + \beta o\{x_2(t)\} = \alpha y_1(t) + \beta y_2(t)$$

- **The system input can be *any* function including an impulse or a *delta function:***

$$y_\delta(t;\tau) = h(t)\delta(t-\tau)$$

Observing an Impulse Response Function (1)

- **Observation 1 – The following symbolic identity holds:**

$$h(t)\delta(t-\tau)=h(\tau)\delta(\tau-t)$$

- **Observation 2 - The product $h(t)\delta(t-\tau)$ is different from zero at the point *t=τ*.**

- **Observation 3 - The impulse response of the system has a circular symmetric property with respect to its arguments *t* and *τ*.**

$$y_\delta(t;\tau)=y_\delta(\tau;t)$$

Observing an Impulse Response Function (2)

- **Observation 4 – In the *(t,τ)-plane*, due to the circular symmetry and because delta function is an even function, we can define a bivariate response function as:**

$$y(t;\tau)=h(t,\tau)x(\tau,t)\delta(|t-\tau|)$$

- **Observation 5 - The ordinary output response at the point *t=τ is:***

$$y(\tau)=\int_{0_-}^{+\infty} h(t,\tau)x(\tau,t)\delta(|t-\tau|)dt=h(\tau)x(\tau)$$

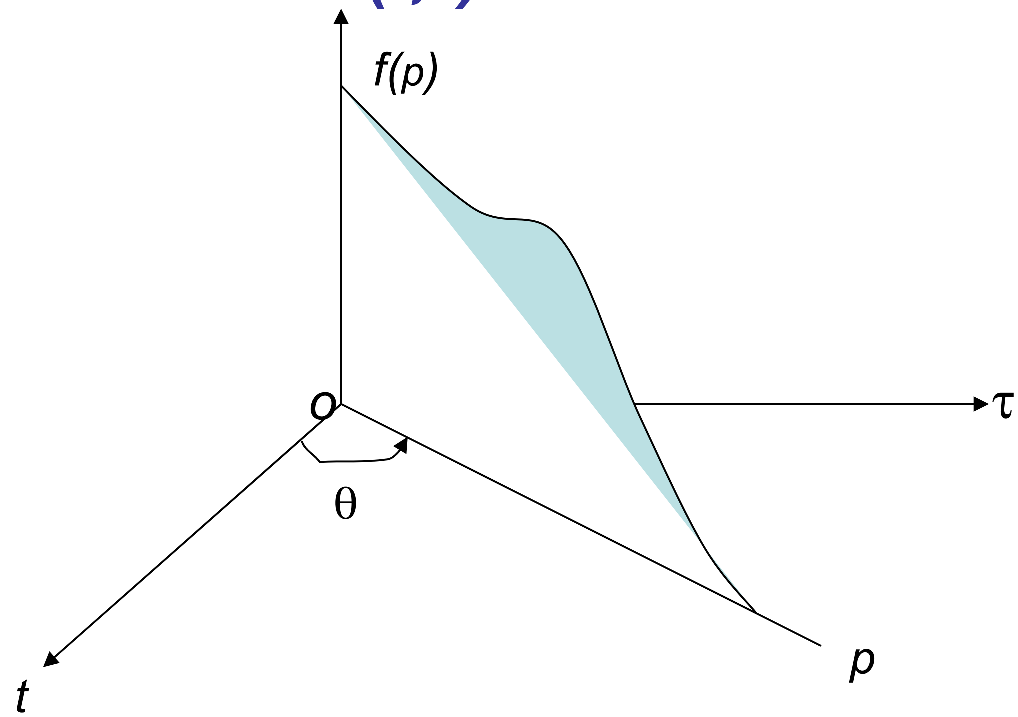
or, equivalently:

$$y(t)=\int_{0_-}^{+\infty} h(t,\tau)x(\tau,t)\delta(|\tau-t|)d\tau=h(t)x(t)$$

Question – Can a system function be equal to an input function?

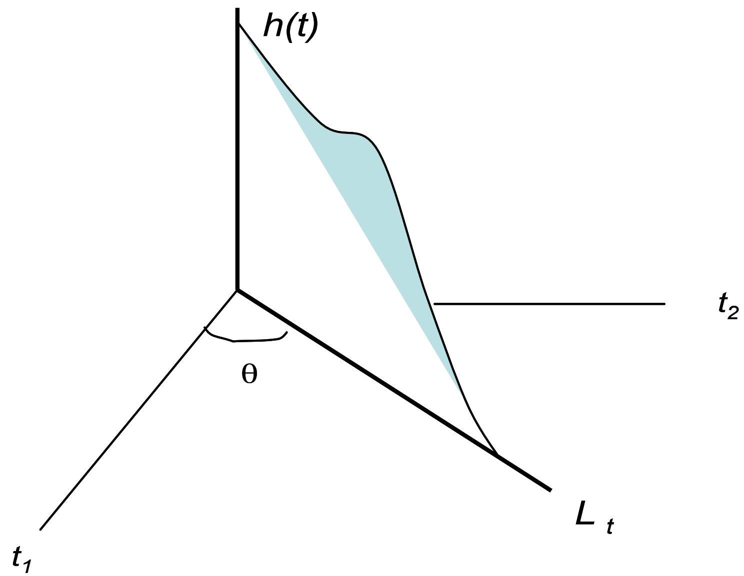
Is the system output considered to be still linear?

Real-Variable Function Representation in (t,τ)-Plane



- A system function, which has the simple rotational property of circular symmetry is shown in this figure.
- Figure shows conversion of a one-dimensional profile of a system function of p to a two-dimensional function of a complex variable of t and τ.
- The common views of the independent signal and system functions are the projection over *t-axis* (pure real) and *τ-axis* (pure imaginary), respectively.

Circularly Symmetric Functions (1)



$$h(t,\theta) = h(t\cos\theta, t\sin\theta) \equiv h_{sym}(t) = \frac{1}{2}[h(t_1,t_2) + h(t_2,t_1)]$$

$$H(s,\phi) \equiv H_{sym}(s) = \frac{1}{2}[H(s_1,s_2) + H(s_2,s_1)]$$

Rotational property of a circularly symmetric function

Impulse-Response System Representation

- **The system response for a circularly symmetric system function can equivalently be written as:**

$$y_\delta(t,\tau) = h(t,\tau)\delta(t-\tau)$$

- **The more familiar impulse response, using *sifting property* of the *delta function* will be:**

$$y_\delta(\tau) = \int_{t=\tau_-}^{t=\tau_+} h(t,\tau)\delta(t-\tau)dt = h(\tau)$$

> **The limits of integration can be extended to infinity.**

Laplace Transform of the Impulse Function

- The ordinary unilateral Laplace transform of $\delta(t-\tau)$ is obtained as:

$$L\{\delta(t-\tau)\}=\int_{0-}^{+\infty}\delta(t-\tau)e^{-s_1 t}dt=e^{-s_1\tau}$$

- This is a function of the variable time *τ*.

- A second transformation yields:

2DLT

$$L_{2D}\{\delta(t-\tau)\}=\int_{0-}^{+\infty}e^{-s_1\tau}e^{-s_2\tau}d\tau=\frac{1}{s_1+s_2}$$

Two-Dimensional Laplace Transform (2DLT)

Definition of 2DLT – The ordinary unilateral 2DLT is defined as:

$$\boldsymbol{H(s_1, s_2) = \int_0^{+\infty}\int_0^{+\infty} h(t_1, t_2) e^{-s_1 t_1} e^{-s_2 t_2} dt_1 dt_2}$$

Inverse Transformation - The inverse 2DLT is given by:

$$L_{2D}^{-1}\{H(s_1, s_2)\} = h(t_1, t_2)$$

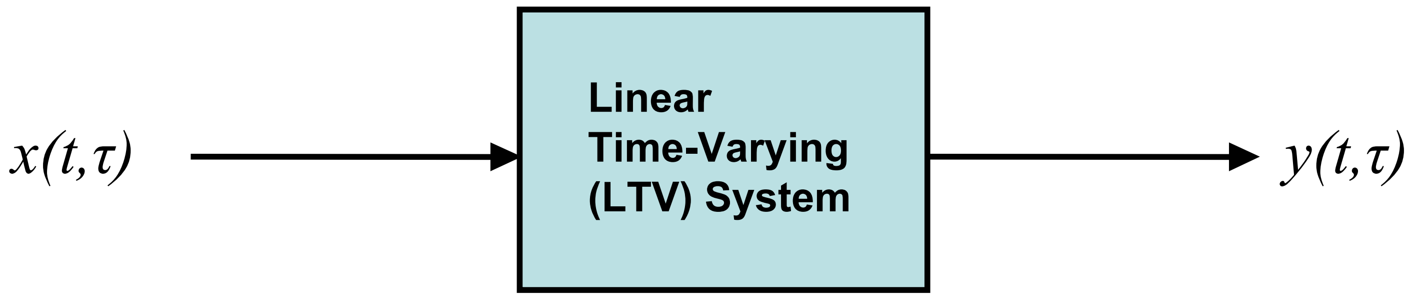
$$= \frac{1}{(2\pi j)^2}\int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} H(s_1, s_2) e^{s_1 t_1} e^{s_2 t_2} ds_1 ds_2$$

Integral Representation

- **Consider a SISO linear dynamic system of finite order characterized by its fundamental equation:**

$$y(t)=\int_{-\infty}^{+\infty} h(t,\tau)x(\tau)d\tau$$

  - **Where *h(t, τ)* is the system function defines the response in the first-quadrant of a rectangular coordinates system.**



Observations on LTV Systems

- **Mathematically speaking, *t* and *τ* represent time variables of an applied signal and the corresponding system.**
- **Variations of an input signal and an autonomous system are independent of each other.**
- **For circularly symmetrical systems, with no loss of generality, we can rewrite the response as:**

$$y(t,\tau) = e^{-\ln h(t,\tau)} x(t)$$

Generalized-Delay System Representation

- If *h(t)* is a (piecewise) continuous function and bounded by a finite number, its 1st -order and higher-order derivatives exists.

- The system response can equivalently be written as:

$$y(t) = e^{-\int_{\tau}^{t} \frac{h'(\xi)d\xi}{h(\xi)}} x(t)$$

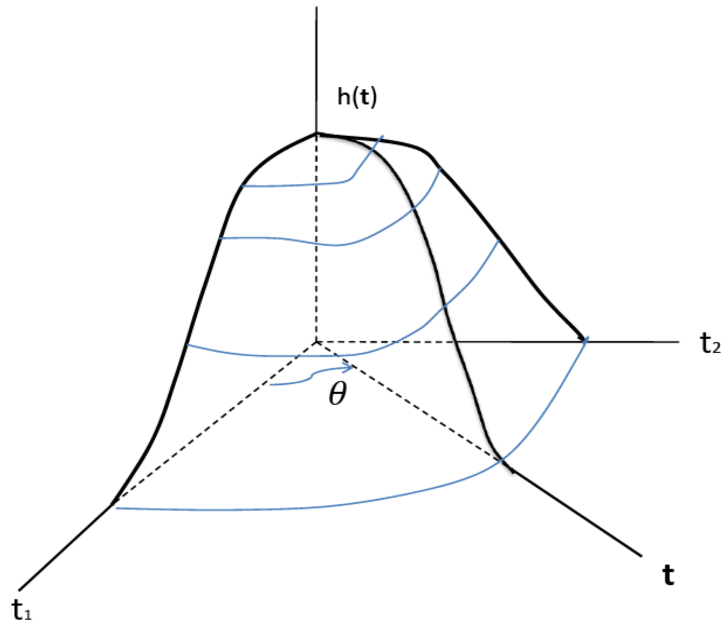
- The system response can be written more compactly as a generalized-delay operator:

$$y(t) = e^{-g(t,\tau)} x(t)$$

Circularly Symmetric Functions (2)

- **From an operational point of view, a circularly symmetric system function *h(t, τ)* can be written as:**

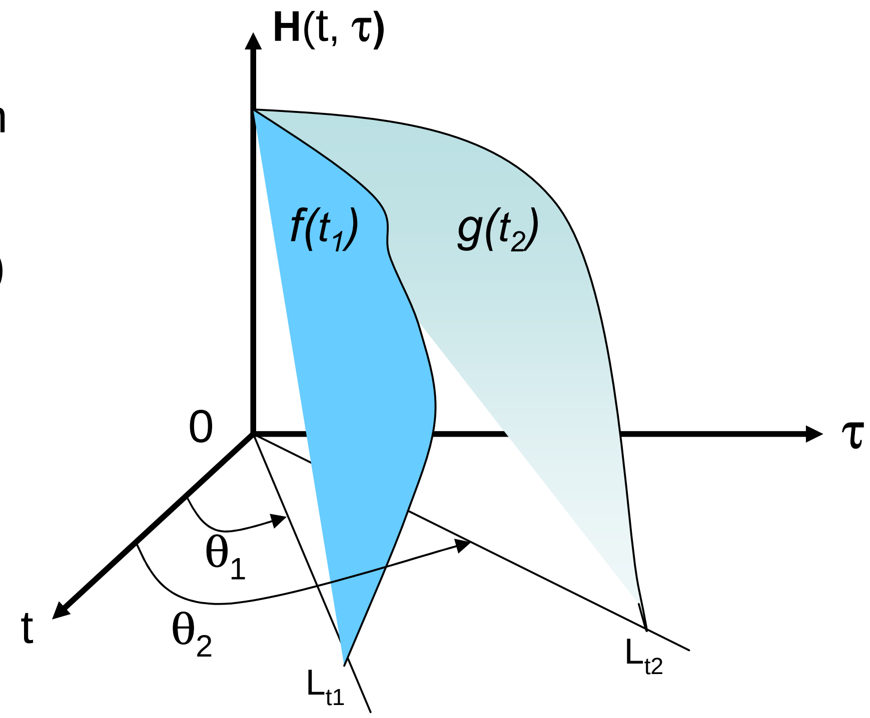
$$h(t,\tau) = h(\sqrt{t^2+\tau^2})$$



Bivariate-Time and Bifrequency

Observation 6 - A general system-theoretic approach for characterization of linear time-varying (LTV) systems based on the application of a two-dimensional Laplace transform (2DLT) is feasible.

Observation 7 - This technique appears to have remained largely unknown to the analog signal processing community up to now.



$$\mathrm{L}\{f(t)\} = F(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$$

Vectors: $\mathbf{s} = [s_1, s_2]$

$t = [t_1, t_2]$

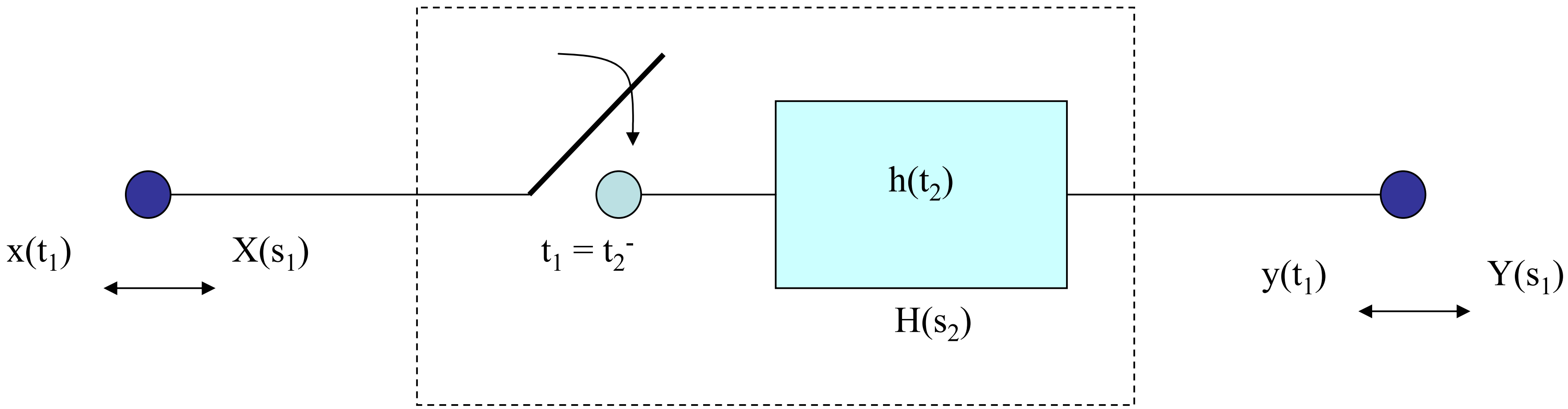
Two *convolving* real-variable function representations in the *time*-plane (in polar coordinates)

Fundamental Input-Output Representation

- **The classical theory of variable systems is based on the solutions of linear ordinary differential equations with varying coefficients.**
  - **The varying coefficients are functions of an independent variable, conveniently called the *time*.**
  - **The *time* is assumed to be *real* for physical systems.**

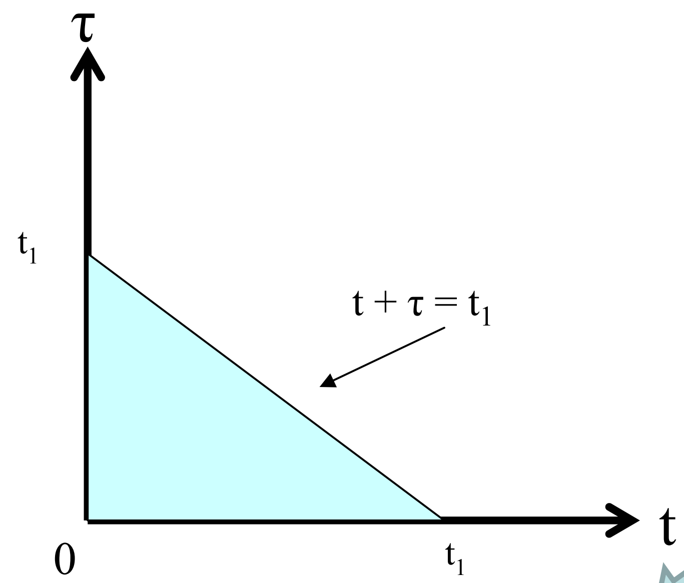
$$\sum_{i=0}^{n} a_i(t)\, y^{(i)}(t) = \sum_{k=0}^{m} b_k(t)\, x^{(k)}(t)$$

Bifrequency Input-Output System Representation



Black box representation of circularly symmetric linear systems

Circularly Symmetric System Transformation



Iterated Laplace Transform

$$Y(s) = \iint_{\Delta} h(t,\tau)x(\tau)e^{-s(t+\tau)}dt\,d\tau$$

Region over which the impulse-response *h(t, τ)* of a nonanticipative system is defined is the shaded area.

Time-Domain Representation of Nonanticipative System Functions

- **Define the instant at which the input is applied to the system as the origin for time *"t."***

- **The nonanticipative condition implies:**

$$h(t-\tau)x(t)u(t) \equiv o \qquad for \quad t < \tau$$

$$h(t-\tau)x(t)u(t) \equiv y(t,\tau) \qquad for \quad t > \tau$$

- **Then, we may define *h(.)* to be zero for negative values of its argument:**

$$y_\delta(t,\tau) = h(|\tau - t|)\delta(t)$$

Frequency-Domain Representation of Nonanticipative System Functions

- **Let us define:**

$$h_1(t,\tau) = \begin{cases} h(t-\tau) & for \quad t > \tau \\ 0 & for \quad t < \tau \end{cases}$$

- **The 2DLT is:**

$$H_1(s_1,s_2) = \int_0^{+\infty} e^{-s_2\tau} d\tau \int_\tau^{+\infty} e^{-s_1 t} h(t-\tau) dt = H(s_1) \Big/ s_1 + s_2$$

- **Similarly, we define :**

$$h_2(t,\tau) = \begin{cases} h(\tau-t) & for \quad \tau > t \\ 0 & for \quad \tau < t \end{cases}$$

$$H_2(s_1,s_2) = H(s_2) \Big/ s_1 + s_2$$

2DLT

- **Adding together, we obtain:**

$$L_{2D}\{h(|\,t-\tau\,|)\} = H(s_1,s_2) = \frac{H(s_1) + H(s_2)}{s_1 + s_2}$$

The 2DLT of General LTV Systems

Consider a SISO LTV system, initially at rest, described by:

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t)}{dt^i} = \sum_{k=0}^{m} b_k(t)\frac{d^k x(t)}{dt^k}$$

An input *x(.)* is applied to the system at time $\xi = t - \tau$

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t,\tau)}{dt^i} = \sum_{k=0}^{m} b_k(t)\frac{d^k x(t-\tau)}{dt^k}$$

This may demands more initial conditions that the problem requires!

Taking a 2DLT, we obtain:

$$\sum_{i=0}^{n}\int_0^{\infty}\int_0^{\infty} a_i(t)\frac{d^i y(t,\tau)}{dt^i}e^{-s_2 t}e^{-s_1\tau}dt\,d\tau = \sum_{k=0}^{m}\int_0^{\infty}\int_{\xi=\tau_{-}}^{t} b_k(t)\frac{d^k x(t-\tau)}{dt^k}e^{-s_2 t}e^{-s_1\tau}dt\,d\tau$$

More on the 2DLT

- **Observation 9** – The 2DLT of a circularly symmetric function with the property $\lim_{t \to \infty} h(t) \to 0$ that is a Hankel transform of order zero. Hankel transforms are integral transformations whose kernels are Bessel functions.

  - **The bilateral 2DLT of a LTV resistor $h(t) = 1/t$ in the frequency domain is $1/\omega$.**
  - **The bilateral 2DLT of a circular unit-step $u(a-t)$, whose value is equal to unity for $|t| < a$, in the frequency domain is $aJ_1(a\omega)/\omega$.**

- **Observation 10** – A change of time-scale $t \to -\ln t$ will transform the 2DLT of a circularly symmetric function into a Mellin transform.

The Hankel Transform

- The Hankel transform is compatible with LTV systems described by a general **Bessel** equation given as:

$$\left[\frac{d^2}{dt^2}+\frac{1}{t}\frac{d}{dt}-\left(\frac{n^2}{t^2}\right)\pm a^2\right]^N y(t)=x(t)$$

- The Hankel transform pairs are symmetric because it deals with symmetric functions.
- The 2DLT of a circularly symmetric function with the property $\lim_{t\to\infty} h(t) \to 0$ that is a Hankel transform of order zero.
- This property is quite useful in application of Hankel transforms to LTV systems.

The Mellin Transform

- The Mellin transform is compatible with LTV systems characterized by a general **Euler-Cauchy** equation given as:

$$\sum_{i=0}^{n} a_i t^i \frac{d^i y(t)}{dt^i} = x(t)$$

- The impulse response of this nonanticipative Euler-Cauchy LTV system is:

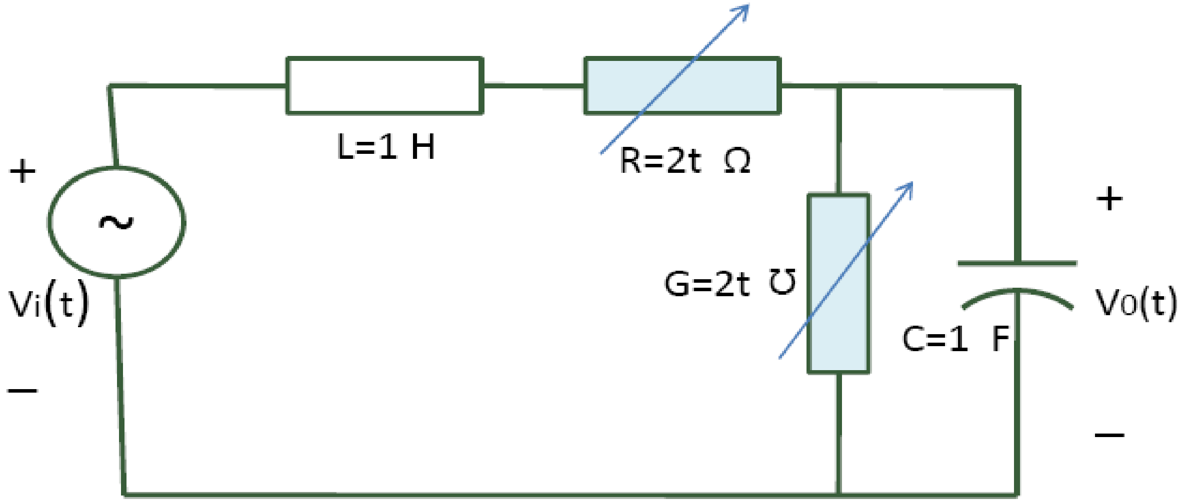
$$h(t,\tau) = \frac{1}{t} g\left(\frac{t}{\tau}\right) u(t-\tau)$$

  - where *g(.)* is the impulse response of a prototype LTI system obtained by changing the time scale $t \rightarrow -\ln t$

- The 2DLT in this case becomes the following Mellin Transform pairs:

$$M\{h(t)\} = \int_{0}^{+\infty} h(t) t^{s-1} dt \qquad M^{-1}\{H(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} H(s) t^{-s} ds$$

An Illustrative Example



The 2DLT of impulse-response is obtained as: $$H(s_1,s_2)=\frac{s_1s_2}{s_1s_2+s_1+2}$$

The inverse function using tables of 2DLT is: $$h(t_1,t_2)=e^{-t_2}J_o(2\sqrt{2t_1t_2})$$

Compare it with the impulse response function for the initially relaxed circuit obtained by using the inductor flux and capacitor charge as state variables:

$$h(t)=e^{-t^2}\ \sin t$$

Conclusions

- **The 2DLT techniques are applicable to LTV systems.**
- **This approach allows, in effect, two-dimensional transform techniques to be used for the time-varying systems in the same manner that the conventional frequency-domain techniques are used in connection with fixed systems.**
- **The 2DLT method applied to an Euler-Cauchy system and a Bessel system results in a Mellin transform and Hankel transform, respectively.**
- **The 2DLT, Mellin transform, and Hankel transform can be derived from the two-dimensional Fourier transform.**
- **The work presented here opens several areas for further investigations in theory of variable systems.**

For Further Information

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Questions?

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Dr. Erfani has published more than 70 technical papers, holds three patents, and is the Senior Technical Editor of the *Journal of Network and Systems Management* and an associate editor for *Computers & Electrical Engineering: An International Journal*.

He received a combined B.Sc. and M.Sc. degree in Electrical Engineering from the University of Tehran in 1971, and M.Sc. and Ph.D. degrees, also in Electrical Engineering, from Southern Methodist University in 1974 and 1976, respectively. He was a Member of Technical Staff at Bell Labs of Lucent Technologies in Holmdel, New Jersey, from 1985 to 2001.